# Transverse Waves in Superfluid $^3$He-B

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We examine the theory of collisionless transverse current waves in bulk superfluid $^3$He-B, including the coupling to the order parameter collective modes. At low frequencies, $\omega \ll \Delta(T)$, the order parameter modes do not contribute to the restoring force for a transverse current, and the quasiparticle contribution drops rapidly as the gap in the spectrum develops. Thus, low-frequency transverse sound becomes overdamped at temperatures near $T_c$. However, at low temperatures ($T \lesssim 0.3\ T_c$) the off-resonant coupling to the $J = 2^-$, $M = \pm 1$ modes stabilizes a propagating transverse current mode, with a large phase velocity and low damping for frequencies above a critical frequency that is approximately that of the $J = 2^-$ mode. We also discuss the similarities and differences of longitudinal and transverse sound in the superfluid phases. For example, in zero field right- and left-circularly polarized waves are degenerate. A magnetic field, with $\vec{H} || \vec{q}$, lifts this degeneracy, giving rise to the analog of circular dichroism and birefringence of electromagnetic waves. Thus, transverse waves may be more easily detected in the B-phase than in normal $^3$He.

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## I. INTRODUCTION

Many predictions of Landau's theory of a normal Fermi liquid have been confirmed by experiments on the low-temperature phases of $^3$He (*c.f.* Baym and Pethick [1]). Most notable is the prediction of a collisionless longitudinal sound mode ('zero sound'). Landau also pointed out the possibility of a collisionless transverse sound mode, *i.e.* a propagating shear wave. [2] While the experimental confirmation of longitudinal zero sound is a well known achievement, [3] the evidence for transverse sound in $^3$He is much less clear. We briefly review past work on transverse sound in $^3$He, and then present theoretical arguments that a propagating transverse current mode may be most easily detected at very low temperatures in the B-phase of superfluid $^3$He.

Ordinary sound in $^3$He propagates as a result of collisions between the quasiparticles, which restore local equilibrium on a timescale $\tau$ that is much shorter than the period of the wave, $\omega\tau \ll 1$. The mean collision time in a normal Fermi liquid becomes very long at low temperatures, $\tau \sim 1/T^2$. The collisional restoring force necessarily breaks down and hydrodynamic sound ceases to propagate. The transition to collisionless sound, for a fixed frequency $\omega$, occurs at a temperature $T_x$ given by $\omega\tau(T_x) \simeq 1$. In this regime sound propagates as a coherent particle-hole excitation of the Fermi sea; however, other propagating modes are possible, including transverse sound.

The restoring force for collisionless sound modes is the Landau molecular field. The velocities of these modes are related to the Landau interaction parameters, $F_0^s$ and $F_1^s$, characterizing the $l = 0$ and $l = 1$ harmonics of the molecular field. The $l = 0$ interaction is related to the compressibility of $^3$He and is large and positive; $F_0^s \simeq 10-100$ for pressures ranging from zero to 34 bar. Similarly, $F_1^s$ determines the molecular field associated with the quasiparticle current, and ranges between $5 - 15$ over the same pressure range. For a collisionless mode to propagate in the normal state, the phase velocity must be greater than the Fermi velocity; otherwise the mode can decay into quasiparticle-quasihole pairs. For longitudinal sound, this condition is obeyed by a considerable margin for all pressures, and primarily reflects the incompressibility of liquid $^3$He. Since transverse sound does not involve density fluctuations, the restoring force is considerably smaller, and is almost entirely due to the molecular field associated with the quasiparticle current. Transverse sound is expected to propagate, albeit with a velocity close to the Fermi velocity, $c_t \simeq 1.2\ v_f$ at $p = 34$ bar. [4]

Observation of transverse zero sound has proven difficult. Roach and Ketterson [5] reported direct transmission of transverse zero sound below 4 mK. However, Flowers, *et al.* [6] pointed out that both incoherent quasiparticle and collective excitations would contribute comparably to the transverse current. In fact the analysis by Flowers *et al.* of the pressure dependence of the attenuation length indicates that the signal reported in ref.[ [5]] was due to single particle excitations and not the collective mode. Thus, to date there is no direct evidence of propagating transverse sound in the normal phase of $^3$He, and even less evidence for a propagating mode below $T_c$. [7] However, transverse acoustic impedance measurements in the normal phase are interpreted in terms of contributions from both a coherent transverse sound mode and an incoherent quasiparticle response. [5,8]

Compared to the literature on longitudinal sound, there have been relatively few theoretical papers on transverse sound in superfluid $^3$He. Initial investigations concluded that the mode, which is already difficult to detect in the normal state, would rapidly disappear below $T_c$. [9–12] In an early paper Maki calculated the transverse current response function in $^3$He-B. [10] Although his result includes the coupling to the squashing mode, Maki analyzed his result for $\omega \ll \Delta$, where the condensate remains in local equilibrium, and obtained the dispersion relation and a criterion for the disappearance of transverse sound. The temperature below which transverse sound would no longer propagate was given by $Y(T_0)F_1^s/6 = 1$, which for $F_1^s = 15$ (high pressure) is roughly, $T_0 \simeq 0.7\ T_c$. [10] Combescot and Combescot [11] came to a similar conclusion regarding the fate of low-frequency transverse sound. The only theoretical work on transverse sound at high frequencies is that of Maki and Ebisawa [13] who examined the coupling of transverse currents to the $J = 2^-$ modes. These authors also emphasize the disappearance of transverse sound below $T_c$, but do not restrict themselves to low frequencies. The numerical calculations reported in ref.[ [13]] are for frequencies below the $J = 2^-$ mode, $\omega < \omega_{2-}(0) \simeq \sqrt{12/5}\Delta_0$; they find a purely reactive response at low temperatures, *i.e.* non-propagating solutions. While some of the important results were obtained early on by the above authors, developments in low-temperature acoustics and our understanding of the collective mode spectrum of superfluid $^3$He make it worthwhile to revisit transverse current modes in superfluid $^3$He.

The disappearance of low-frequency transverse sound in the superfluid phases should be contrasted with a propagating longitudinal sound mode at all temperatures below $T_c$. There are two important differences for low-frequency, collisionless ($1/\tau \ll \omega \ll \Delta(T)$) longitudinal and transverse sound modes in superfluid $^3$He. First, as noted earlier, transverse sound does not couple to density fluctuations, for which there is a very strong restoring force; since the restoring force for transverse current dies out rapidly, so does the mode. Secondly, density and longitudinal current fluctuations couple to the phase mode of the order parameter, which has a phonon dispersion relation. These features, the coupling to the propagating phase mode combined with a large restoring force from the quasiparticle contribution to the molecular field, are responsible for a propagating longitudinal sound mode at all temperatures. In short, transverse current does not propagate at low frequencies because it does not couple to the phase mode, nor to any other propagating mode.

At higher frequencies, $\omega \sim \Delta$, density and current fluctuations couple to other collective modes of the order parameter. These modes are broadly classified into two types: (i) Goldstone modes, associated with a spontaneously broken continuous symmetry, and (ii) exciton modes (also referred to as pair-vibration modes), which correspond to time-dependent deformations of the order parameter. In addition to the phase mode, which is the Goldstone mode associated with broken gauge symmetry, [14,15] there are other Goldstone modes associated with broken rotational symmetries, although they are not important for the discussion of sound propagation in homogeneous $^3$He-B. Exciton modes have been studied extensively in superfluid $^3$He, both experimentally and theoretically [*c.f.* ref.( [16])]. In particular, ultrasound spectroscopy of the order parameter collective modes played an important role in confirming the identification of the A and B phases based on NMR spectroscopy. [16,17]

In the following section we discuss the symmetries of $^3$He-B, the spectrum of collective modes and the selection rules that govern the coupling of various modes in the B-phase. The dispersion relation for transverse current is derived from quasiclassical linear response theory in section (III). We present analytical and numerical results for the complex phase velocity of transverse waves, and show that transverse currents propagate with low attenuation in the frequency range $\omega_{2-}(0) < \omega < 2\Delta_o$, for low temperatures, $T \lesssim 0.3\ T_c$. We briefly discuss the similarities and differences between longitudinal and transverse current modes. Right- and left circularly polarized transverse modes propagate with different complex phase velocities in a longitudinal magnetic field, $\vec{H}||\vec{q}$. Thus, these modes exhibit the analog of circular birefringence and dichroism of electromagnetic waves in a magnetic medium. The rotation of the polarization of a linearly polarized transverse excitation would be a definitive method of detecting a propagating transverse wave.

## II. SYMMETRIES AND SELECTION RULES

The superfluid phases of $^3$He are spin-triplet, p-wave pair condensates [*c.f.* Vollhardt and Wölfle [17]], with an order parameter given by the mean-field self-energy of the pairs,

$$\Delta_{\alpha\beta}(\hat{p}) = \int \frac{d\Omega_{p'}}{4\pi} V(\hat{p}\cdot\hat{p}') \int_{-\omega_c}^{\omega_c} d\xi_{\vec{p}'} \ < a_{\vec{p}'\alpha} \ a_{-\vec{p}'\beta} > \ . \tag{1}$$

The equilibrium B-phase is identified as the Balian-Werthamer (BW) state, represented by the $3\times 3$ complex matrix

$$d_{ij} = \frac{\Delta}{\sqrt{3}} \ R_{ij}[\hat{n},\vartheta] \ e^{i\Phi} \ , \tag{2}$$

which transforms as vector under spin (orbital) rotations with respect to the index $i$ ($j$). In the above equation, $\Delta$ is the amplitude, $\Phi$ is the phase, and $R_{ij}$ is an orthogonal matrix defining the relative orientation of the spin and orbital coordinates of the Cooper pairs. The matrix $d_{ij}$ is related to the spin-triplet, p-wave mean-field self energy by $\Delta_{\alpha\beta}(\hat{p}) = [i\sigma^i\sigma^2]_{\alpha\beta} d_{ij}\hat{p}_j$. The BW state is an eigenfunction of total spin ($S=1$) and total orbital angular momentum ($L=1$). And, although the relative spin-orbit symmetry is spontaneously broken, the BW state is nevertheless an eigenstate of the total angular momentum, defined by $\vec{J} = \vec{L} + \mathbf{R}^{-1}\cdot\vec{S}$, with $J=0$. As a consequence of the isotropy of the BW state, the excitations of the order parameter are also classified by the quantum numbers $(J,M)$, where $M$ is the quantum number for the projection of $\vec{J}$ along a fixed quantization axis defined by the larger of the Zeeman or dispersion energies. [18] For example, excitations of the amplitude, $\Delta$, or the phase, $\Phi$, have the same rotational symmetry as the ground state, and so have quantum number $J=0$. However, more complicated excitations of the order parameter have quantum numbers $J=1$, with $M=0,\pm1$, or $J=2$, with $M=0,\pm1,\pm2$.

The amplitude, $\Delta$, is a *stiff* degree of freedom, fixed by the minimum of the condensation energy. Therefore, the amplitude mode (or any deformation from the local equilibrium form of eq.(2)) has a finite gap at $q\rightarrow 0$. On the other hand, the phase mode and the rotation matrix are *soft* degrees of freedom because the free energy is invariant under uniform gauge transformations and/or uniform spin and orbital rotations. Thus, long wavelength excitations of these soft variables have no gap, but typically a linear dispersion relation for $q\rightarrow 0$. In particular, at $T=0$ the phase mode has the dispersion relation, $\omega = \frac{v_f}{\sqrt{3}}q$, in the long wavelength limit. [14,15] This mode, which strongly couples to the density and longitudinal current, has its velocity pushed up to the hydrodynamic sound velocity, $c_1$ (for $\omega \ll \Delta$), [19] and is responsible for the survival of longitudinal collisionless sound in the superfluid as $T\rightarrow 0$.

In addition to these qualitative features of the collective mode spectrum, there is the doubling of the number of excitations for each rotational quantum number $(J,M)$. This doubling reflects the fact that the general order parameter is a complex $3\times 3$ matrix. Connected with this doubling is the discrete particle-hole symmetry of the normal Fermi liquid. The particle-hole operation transforms a quasiparticle with energy $\xi_{\vec{k}} > 0$ into a quasihole with energy $\xi_{\underline{\vec{k}}} = -\xi_{\vec{k}} < 0$ (with a spin reversal), and is represented by a unitary operator $\mathcal{C}$ that has the action, $\mathcal{C} a_{\vec{k}\alpha}\mathcal{C}^{\dagger} = [i\sigma^2]_{\alpha\beta} \ a^{\dagger}_{\underline{\vec{k}}\beta}$. This operation is an approximate symmetry of the normal Fermi-liquid ground state, and as a consequence the order parameter, $d_{ij}$, transforms into its complex conjugate. [20] Thus, the real and imaginary parts, $d^{\pm}_{ij} = d_{ij} \pm d^{*}_{ij}$, have definite signatures under the particle-hole transformation, $d^{\pm}_{ij} \xrightarrow{\mathcal{C}} \pm \, d^{\pm}_{ij}$. Excitations of the condensate can then be labeled by the set of quantum numbers $\{J,M,\zeta\}$, where $\zeta = \pm$ is the signature under particle-hole interchange. The utility of this description is most evident when we consider selection rules for processes involving these excitations. [20,21]

Although many of the remarkable properties of superfluid $^3$He-B are connected with the Goldstone modes, they are largely irrelevant for propagating transverse current waves. Even the phase mode, which is so important for longitudinal sound, does not couple to transverse current in the isotropic B-phase. However, transverse currents do couple to the $J=2$, $M=\pm1$ modes of the order parameter. Although these modes have excitation energies of order $\Delta$, they provide a contribution to the molecular field which stabilizes the propagation of transverse current waves. This is analogous to the role of the phase mode in the case of longitudinal sound. However, the differences in the dispersion relations for the phase mode and the high-frequency $J=2$ modes leads to more complex frequency and temperature dependences for the transverse mode velocity. Before we analyze the dispersion relation in detail we first discuss the symmetry constraints on the allowed couplings between the density, current and order parameter modes.

Consider the conservation laws for particle number and momentum,

$$\omega\delta n - \vec{q}\cdot\delta\vec{j} = 0 \ , \tag{3}$$

$$\omega\delta j_i - \frac{1}{m_3}\delta\Pi_{ij}q_j = 0 \ . \tag{4}$$

These equations must be supplemented by constitutive equations connecting the fluctuations of the stress tensor, $\delta\Pi_{ij}$, the current, $j_i$, the density, $\delta n$, and the order parameter, $\delta d^{\pm}_{ij}$. Note that for a Galilean invariant system the momentum current is $m_3\ \delta\vec{j}$, where $m_3$ is the atomic mass of $^3$He. [1] In linear response the fluctuation in the stress tensor has the general form,

$$\delta\Pi_{ij} = \mathcal{A}\delta n\ \delta_{ij} + \mathcal{B}(\delta j_i q_j + q_i \delta j_j) + \mathcal{E}(\delta d^{+}_{ij} + \delta d^{+}_{ji}) + \mathcal{F}(\delta d^{-}_{ij} + \delta d^{-}_{ji})\ . \tag{5}$$

The symmetry of the equilibrium state places stringent conditions on the form of the constitutive equation. In fact, we have required $\delta\Pi_{ij}$ to be symmetric, and have imposed the rotational invariance of the ground state in writing eq.(5). There is an additional constraint imposed by particle-hole symmetry; the stress tensor transforms as $\delta\Pi_{ij} \xrightarrow{\mathcal{C}} -\delta\Pi_{ij}$, as does the density, current and imaginary part of the order parameter. Thus, for eq.(5) to be consistent with particle-hole symmetry the coupling to the real part of the order parameter, $\delta d^{+}_{ij}$, must vanish. Of course, the fact that particle-hole symmetry is weakly broken (at the level $\mathcal{O}(T/E_f)$) implies a small, but non-vanishing, coupling to the real order parameter modes; [22] it is this coupling that is responsible for the sharp resonance observed in the longitudinal sound attenuation at the frequency of the $J = 2^+$ modes. [23–25]

Here we are interested in the coupling of the order parameter to transverse current. It is then convenient to represent the current as,

$$\delta\vec{j}(\vec{q},\omega) = \sum_{\mu=0,\pm} \delta j_\mu \hat{e}^{(\mu)}\ , \tag{6}$$

with $\hat{e}^{(0)} = \hat{q}$ defining the propagation direction, and $\hat{e}^{(\pm)} = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$ for the transverse circular polarizations. The order parameter fluctuations may also be written in terms of these and related elementary tensors,

$$\delta d^{\pm}_{ij} = \sum_{J,M} \mathcal{D}^{\pm}_{JM}(\vec{q},\omega)\ t^{(J,M)}_{ij}\ , \tag{7}$$

where $\{t^{(J,M)}_{ij}\}$ are spherical tensors obeying the orthogonality conditions, $tr[\hat{t}^{(J,M)}\hat{t}^{(J',M')\dagger}] = \delta_{J,J'}\delta_{M,M'}$. The basis used here is [26]

$$t^{(0,0)}_{ij} = \frac{1}{\sqrt{3}}\delta_{ij}\ , \tag{8}$$

$$t^{(1,M)}_{ij} = \frac{1}{\sqrt{2}}\varepsilon_{ijk}\ \hat{e}^{(M)}_k\ , \tag{9}$$

$$t^{(2,0)}_{ij} = \sqrt{\frac{3}{2}}(\hat{e}^{(0)}_i\hat{e}^{(0)}_j - \frac{1}{3}\delta_{ij})\ , \tag{10}$$

$$t^{(2,\pm1)}_{ij} = \frac{1}{\sqrt{2}}(\hat{e}^{(0)}_i\hat{e}^{(\pm)}_j + \hat{e}^{(\pm)}_i\hat{e}^{(0)}_j)\ , \tag{11}$$

$$t^{(2,\pm2)}_{ij} = \hat{e}^{(\pm)}_i\hat{e}^{(\pm)}_j\ . \tag{12}$$

By inspection of eqs.(4)-(12), it is clear that neither longitudinal, nor transverse current couples to the $J = 1$ modes. Also note that the longitudinal sound couples only to the order parameter modes with $J = 0^{\pm}$ and $J = 2^{\pm}, M = 0$. The $J = 0^-$ mode is the phase mode; the $J = 0^+$ mode is the amplitude mode, which does not couple if we enforce particle-hole symmetry. The $J = 2^{\pm}$ modes are the well-studied real and imaginary squashing modes. The coupling of the order parameter to transverse current is also evident from eqs.(5)-(12); only the $J = 2^{\pm}, M = \pm 1$ modes couple to transverse current, and the real modes are eliminated if we enforce particle-hole symmetry. In the next section we calculate this coupling and investigate the conditions for propagating transverse current modes.

## III. LINEAR RESPONSE THEORY

The collective excitations of superfluid $^3$He involve molecular fields, normally associated with particle-hole excitations, as well as particle-particle excitations of the condensate. Indeed these two channels are coupled as a result of pair condensation. Thus, a theory of collective modes in superfluid $^3$He requires a transport equation for the quasiparticle excitations, and a time-dependent gap equation to describe the dynamics of the order parameter. These equations must be supplemented with initial conditions, which we take to be the homogeneous equilibrium B-phase, represented by the order parameter in eq.(2), and its associated quasiparticle spectrum, $N(\epsilon)$, and thermal distribution.

We start from the quasiclassical formulation of non-equilibrium superconductivity, [27–29] and follow closely the notation in ref.[ [30]]. This theory is formulated in terms of a $4\times 4$ matrix propagator in combined particle-hole/spin space,

$$\hat{g}(\hat{p},\epsilon;\vec{q},\omega) = \begin{pmatrix} g & f \\ \bar{f} & \bar{g} \end{pmatrix} , \tag{13}$$

where each entry is a $2\times 2$ spin matrix. The diagonal components are closely related to the distribution functions for Bogoliubov quasiparticles, while the off-diagonal elements are the Cooper pair amplitudes. There is some redundancy in this description; the functions $\bar{f}$ and $\bar{g}$ satisfy the identities,

$$\bar{f} = -f(-\hat{p},-\epsilon;-\vec{q},-\omega)^* \qquad , \qquad \bar{g} = g(-\hat{p},-\epsilon;\vec{q},\omega)^{tr} , \tag{14}$$

as do the diagonal and off-diagonal self energies. In the collisionless, limit the linearized transport equation is

$$\begin{aligned} &(\epsilon+\omega/2)\ \hat{\tau}_3\ \delta\hat{g} - \delta\hat{g}\ (\epsilon-\omega/2)\ \hat{\tau}_3 + \delta\hat{g}\hat{\Delta} - \hat{\Delta}\delta\hat{g} + i\vec{v}_f(\hat{p})\cdot\vec{\nabla}\delta\hat{g} \\ &+\hat{g}_{eq}(\epsilon+\omega/2)\ [\hat{\varepsilon}+\delta\hat{\Delta}] - [\hat{\varepsilon}+\delta\hat{\Delta}]\ \hat{g}_{eq}(\epsilon-\omega/2) = 0 , \end{aligned} \tag{15}$$

where $\hat{\Delta}(\hat{p})$ is the equilibrium order parameter,

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta(\hat{p}) \\ -\Delta(\hat{p})^\dagger & 0 \end{pmatrix} , \tag{16}$$

with $\Delta(\hat{p})$ given by the BW state in eq.(2), and $\hat{\varepsilon}(\hat{p};\vec{q},\omega)$ represents the perturbation in the Landau molecular field (defined below). The corresponding equilibrium propagator is

$$\hat{g}_{eq}(\epsilon) = \alpha(\epsilon)\ \hat{\tau}_3 + \beta(\epsilon)\ \hat{\Delta} , \tag{17}$$

where $\alpha = -\epsilon\ \beta = -2\pi i\ N(\epsilon)\ \tanh(\epsilon/2T)$, and

$$N(\epsilon) = \frac{|\epsilon|}{\sqrt{\epsilon^2-\Delta^2}}\Theta(\epsilon^2-\Delta^2) \tag{18}$$

is the quasiparticle density of states in units of the normal-state density of states at the Fermi surface, $N_f$. The fluctuations of the order parameter,

$$\delta\hat{\Delta}(\hat{p};\vec{q},\omega) = \begin{pmatrix} 0 & \delta\vec{d}\cdot i\vec{\sigma}\sigma_2 \\ \delta\vec{d}'\cdot i\sigma_2\vec{\sigma} & 0 \end{pmatrix} , \tag{19}$$

are determined self-consistently by the weak-coupling gap equation, which we write in terms of the real and imaginary parts of the order parameter fluctuations, $\delta\vec{d}^{\pm} = \delta\vec{d} \pm \delta\vec{d}'$ (we use the same convention for other variables),

$$\delta\vec{d}^{\pm} = \int \frac{d\Omega'}{4\pi}\ V(\hat{p}\cdot\hat{p}') \int_{-\omega_c}^{\omega_c} \frac{d\epsilon}{4\pi i}\ \delta\vec{f}^{\pm}(\hat{p},\epsilon;\vec{q},\omega) . \tag{20}$$

The pairing interaction is taken to be pure p-wave with a cut-off ($T_c \ll \omega_c \ll E_f$), which is replaced in favor of the transition temperature via the linearized gap equation, $\frac{1}{V_1} = \int_{-\omega_c}^{\omega_c} d\epsilon\ \frac{\tanh(\epsilon/2T_c)}{2\epsilon} \simeq \ln(1.13\omega_c/T_c)$.

The diagonal mean field, $\hat{\varepsilon}(\hat{p};\vec{q},\omega)$, represents the fluctuation in the Landau molecular field, and is determined self-consistently in terms of the diagonal quasiclassical propagator,

$$\hat{\varepsilon}(\hat{p};\vec{q},\omega) = \begin{pmatrix} \varepsilon(\hat{p}) + \vec{h}(\hat{p})\cdot\vec{\sigma} & 0 \\ 0 & \varepsilon(-\hat{p}) + \vec{h}(-\hat{p})\cdot\vec{\sigma}^{tr} \end{pmatrix} , \tag{21}$$

where the scalar and spin-vector fields are,

$$\varepsilon(\hat{p};\vec{q},\omega) = \frac{1}{2}\int \frac{d\Omega'}{4\pi}\ A^s(\hat{p}\cdot\hat{p}')\ \delta\breve{g}(\hat{p}';\vec{q},\omega) , \tag{22}$$

$$\vec{h}(\hat{p};\vec{q},\omega) = \frac{1}{2}\int \frac{d\Omega'}{4\pi}\ A^a(\hat{p}\cdot\hat{p}')\ \delta\breve{\vec{g}}(\hat{p}';\vec{q},\omega) + \vec{h}_{ext} , \tag{23}$$

$A^{s,a}(x) = \sum_{l\geq 0}[F_l^{s,a}/(1+F_l^{s,a}/2l+1)]P_l(x)$ are the Fermi-liquid interactions in the scalar and spin channels, and $\vec{h}_{ext} = -\frac{\gamma}{2}\vec{H}/(1+F_0^a)$ is the external field. We have introduced the $\epsilon$-integrated propagator,

$$\delta\breve{g}(\hat{p};\vec{q},\omega) = \int \frac{d\epsilon}{2\pi i}\ \delta g(\hat{p},\epsilon;\vec{q},\omega)\ , \tag{24}$$

and a similar notation for other components of $\delta\hat{g}$.

Consider the BW state in zero field, $\vec{h} = 0$. Because of the isotropy of the B-phase only the low-order Landau parameters, $F_0^s$, $F_1^s$, and $F_2^s$ turn out to be relevant. The diagonal mean field is then represented by the even- and odd-parity terms,

$$\hat{\varepsilon}(\hat{p};\vec{q},\omega) = \frac{1}{2}\Big\{\ A_0^s\ \delta g_0\ +\ \frac{A_2^s}{5}\sum_{M=-2}^{+2}\delta g_{2M}\ (\hat{p}_i t_{ij}^{(2,M)}\hat{p}_j)\Big\}\ \hat{1} + \frac{1}{2}\Big\{\ \frac{A_1^s}{3}\sum_{M=-1}^{+1}\delta g_{1M}(\hat{e}^{(M)}\cdot\hat{p})\Big\}\ \hat{\tau}_3\ , \tag{25}$$

which we have written in terms of the spherical harmonic components of the diagonal propagator,

$$\delta g_0 = \int \frac{d\Omega}{4\pi}\ \delta\breve{g}(\hat{p})\ , \tag{26}$$

$$\delta g_{1M} = 3\int \frac{d\Omega}{4\pi}\ (\hat{e}^{(M)*}\cdot\hat{p})\ \delta\breve{g}(\hat{p})\ , \tag{27}$$

$$\delta g_{2M} = \frac{15}{2}\int \frac{d\Omega}{4\pi}\ (\hat{p}_i t_{ij}^{(2,M)*}\hat{p}_j)\ \delta\breve{g}(\hat{p})\ . \tag{28}$$

Note that the linear combinations $\delta\breve{g}^{\pm}$ and $\varepsilon^{\pm}$ correspond to the even- and odd-parity parts of $\delta\breve{g}$ and $\varepsilon$.

Finally, the quasiclassical propagator is related to the density and current response by, [29]

$$\delta n(\vec{q},\omega) = \frac{N_f}{1+F_0^s}\int \frac{d\Omega}{4\pi}\ \delta\breve{g}(\hat{p}) = \frac{N_f}{1+F_0^s}\ \delta g_0\ , \tag{29}$$

$$\delta\vec{j}(\vec{q},\omega) = N_f\int \frac{d\Omega}{4\pi}\vec{v}_f(\hat{p})\ \delta\breve{g}(\hat{p})\ . \tag{30}$$

The density is proportional to the $l = 0$ moment of the distribution function, while the longitudinal and transverse components of the current are given by the $l = 1$ projections of $\delta\breve{g}$,

$$\delta j_\mu = \frac{1}{3}N_f\ v_f\ \delta g_{1\mu} \qquad ; \qquad \mu = 0, \pm 1\ . \tag{31}$$

An important equation which follows from direct integration of the transport equation for the diagonal propagator is, [21]

$$\Big(\omega - \vec{q}\cdot\vec{v}_f(\hat{p})\Big)\ \delta\breve{g}(\hat{p}) - 2\omega\ \delta\varepsilon(\hat{p}) = 0\ . \tag{32}$$

This equation contains both conservation laws. For example, the $l = 0$ projection of eq.(32) gives

$$\omega(1 - A_0^s)\ \delta g_0 - \frac{1}{3}qv_f\ \delta g_{10} = 0\ , \tag{33}$$

which upon comparison with eqs.(29) and (31) is the continuity equation (3). Similarly, the $l = 1$ projections yield the conservation law for momentum, with the stress tensor identified as

$$\delta\Pi_{ij} = N_f p_f v_f\ \int \frac{d\Omega}{4\pi}\hat{p}_i\hat{p}_j\ \delta\breve{g}(\hat{p})\ . \tag{34}$$

### A. Transverse Waves

The dispersion relation for transverse current is obtained as the solution to (eq.(4)),

$$\left(\frac{\omega}{qv_f}\right)_\pm = \frac{2}{5}\left(1+\frac{F_1^s}{3}\right)\left\{\frac{1}{\sqrt{2}}\frac{\delta g_{2\pm}}{\delta g_{1\pm}}\right\} \ , \tag{35}$$

where the response function $\{\delta g_{2\mu}/\delta g_{1\mu}\}$ is the internal stress, or 'restoring force', generated by a field that couples to transverse current with circular polarization $\mu = \pm$. The restoring force is calculated from the solutions to the linearized transport equations. The relevant matrix elements are given in ref.( [30]), from which we construct the solutions,

$$\delta\breve{g}^+(\hat{p}) = 2\left(\frac{\omega}{\omega^2-\eta^2}\right)(1-\lambda)[\omega\varepsilon^+ + \eta\varepsilon^-] + \omega\bar{\lambda}(\vec{\Delta}\cdot\vec{d}^-) \ , \tag{36}$$

$$\begin{aligned}\vec{d}^-(\hat{p}) = \tfrac{1}{2}\int \tfrac{d\Omega'}{4\pi}\ V(\hat{p}\cdot\hat{p}')\Big\{ &-\vec{\Delta}(\hat{p}')\ \bar{\lambda}(\hat{p}')\ [\omega\varepsilon^+(\hat{p}') + \eta'\varepsilon^-(\hat{p}')] \\ &+[\tfrac{2}{V_1} + \tfrac{1}{2}(\omega^2 - \eta'^2 - 4\Delta^2)\bar{\lambda}(\hat{p}')]\vec{d}^-(\hat{p}') + 2\bar{\lambda}(\hat{p}')[\vec{\Delta}(\hat{p}')\cdot\vec{d}^-(\hat{p}')]\ \vec{\Delta}(\hat{p}')\ \Big\} \ ,\end{aligned} \tag{37}$$

where $\eta = \vec{q}\cdot\vec{v}_{\hat{p}}$. The particular linear combinations of mean fields which enter eqs.(36) and (37) are dictated by the basic symmetries of the quasiclassical propagator and transport equation under Galilean and gauge transformations. [21] The $l = 2$, $\mu = \pm$ components of the diagonal propagator are,

$$\delta g_{2\pm}\Big\{1 - \frac{A_2^s}{5}\ \xi_1(q,\omega)\Big\} = \frac{A_1^s}{3}\Big(\frac{qv_f}{\omega}\Big)\xi_1(q,\omega)\Big(\frac{1}{\sqrt{2}}\delta g_{1\pm}\Big) + \Big(\frac{\omega}{2\Delta}\Big)\Lambda_1(q,\omega)\ \mathcal{D}^-_{2\pm} \ , \tag{38}$$

which exhibits both the direct coupling to the transverse current, via the quasiparticle contribution to the molecular field, as well as the coupling to the order parameter modes, $\mathcal{D}^-_{2\pm}$, which are driven by the transverse current. In addition, there is a Fermi-liquid renormalization of the stress, represented by $A_2^s$. The functions $\xi_1(q,\omega)$ and $\Lambda_1(q,\omega)$ are given by,

$$\xi_n(q,\omega) = 15\int\frac{d\Omega}{4\pi}\ \Big(\frac{\omega^2}{\omega^2 - (\vec{q}\cdot\vec{v}_f(\hat{p}))^2}\Big)\,[1-\lambda(\hat{p})]\ \ |\hat{e}^{(+)}\cdot\hat{p}|^2\ |\hat{p}\cdot\hat{q}|^{2n} \ , \tag{39}$$

$$\Lambda_n(q,\omega) = 15\int\frac{d\Omega}{4\pi}\ \lambda(\hat{p})\ |\hat{e}^{(+)}\cdot\hat{p}|^2\ |\hat{p}\cdot\hat{q}|^{2n} = \frac{15}{2}[\lambda_n - \lambda_{n+1}] \ , \tag{40}$$

$$\lambda_n(q,\omega) = \int\frac{d\Omega}{4\pi}\ \lambda(\hat{p})\ |\hat{p}\cdot\hat{q}|^{2n} \ , \tag{41}$$

where $\lambda = \Delta^2\bar{\lambda}$ is the full $\vec{q}$ and $\omega$-dependent Tsuneto function [see the appendix].

To complete the calculation of the restoring force we solve for the order parameter response to a transverse current, $(\frac{\mathcal{D}^-_{2\mu}}{\delta g_{1\mu}})$, from the inhomogeneous gap equation (37). We first carry out the p-wave projection to obtain,

$$\begin{aligned}&\Big(\frac{\omega^2 - 4\Delta^2}{4\Delta^2}\Big)\ \delta d_{ik}\ I^{(1)}_{kj} - \frac{(qv_f)^2}{4\Delta^2}\ \delta d_{ik}\ I^{(2)}_{kj} + \delta d_{kl}\ I^{(3)}_{klij} = \Big(\frac{\omega}{2\Delta}\Big)A_0^s\ \delta g_0\ I^{(1)}_{ij} \\ &+\Big(\frac{\omega}{2\Delta}\Big)\frac{A_2^s}{5}\sum_M\ \delta g_{2M}\ t^{(2,M)}_{kl}\ I^{(3)}_{klij} + \Big(\frac{qv_f}{2\Delta}\Big)\frac{A_1^s}{3}\sum_M\ \delta g_{1M}\ (\hat{e}^{(0)}_k\hat{e}^{(M)}_l)\ I^{(3)}_{klij} \ ,\end{aligned} \tag{42}$$

where

$$I^{(1)}_{ij} = \int\frac{d\Omega}{4\pi}\ \lambda(\hat{p})\ \hat{p}_i\hat{p}_j \ , \tag{43}$$

$$I^{(2)}_{ij} = \int\frac{d\Omega}{4\pi}\ \lambda(\hat{p})\ (\hat{p}\cdot\hat{q})^2\ \hat{p}_i\hat{p}_j \ , \tag{44}$$

$$I^{(3)}_{ijkl} = \int\frac{d\Omega}{4\pi}\ \lambda(\hat{p})\ \hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l \ , \tag{45}$$

are all cylindrically symmetric tensors which are easily evaluated in terms of the moments of $\lambda(\hat{p})$ in eq.(41). Contracting eq.(42) with the $J = 2^-$, $M = \pm 1$ tensor gives the equation of motion for $\mathcal{D}^-_{2\pm}$,

$$\mathcal{D}^-_{2\pm} = \frac{32}{15}\Delta^2\Big(\frac{\Lambda_1}{\lambda_0+\lambda_1}\Big)\frac{1}{D(q,\omega)}\ \Big[\frac{\omega}{2\Delta}\frac{A^s_2}{5}\ \delta g_{2\pm}\ +\ \frac{qv_f}{2\Delta}\frac{A^s_1}{3}\Big(\frac{1}{\sqrt{2}}\ \delta g_{1\pm}\Big)\Big]\ , \tag{46}$$

where

$$D(q,\omega) = \Big\{\omega^2 - 4\Delta^2\Big(1 - 4\frac{\lambda_1-\lambda_2}{\lambda_0+\lambda_1}\Big) - q^2 v_f^2\Big(\frac{\lambda_1+\lambda_2}{\lambda_0+\lambda_1}\Big)\Big\}\ . \tag{47}$$

Finally, combining eqs.(35), (38) and (46), gives the complete dispersion equation for collisionless transverse currents in zero field,

$$\Big(\frac{\omega}{qv_f}\Big)^2 = \frac{F^s_1}{15}\ \frac{(1+F^s_2/5)\ \xi_1}{1+(1-\xi_1)F^s_2/5}\ +\ \frac{8F^s_1}{225}\ \frac{\Lambda_1^2}{\lambda_0+\lambda_1}\ \frac{(1+F^s_2/5)^2}{(1+(1-\xi_1)F^s_2/5)}\ \frac{\omega^2}{\mathbf{D}(q,\omega)}\ , \tag{48}$$

where the denominator is

$$\mathbf{D}(q,\omega) = \{1+\frac{F^s_2}{5}(1-\xi_1)\}\ D(q,\omega)\ -\ \frac{8}{15}\frac{\Lambda_1^2}{\lambda_0+\lambda_1}\ \frac{F^s_2}{5}\ \omega^2\ . \tag{49}$$

The dispersion equation has two contributions: a quasiparticle term and the contribution from the coupling of the transverse current to the order parameter mode, $\mathcal{D}^-_{2\pm}$. Also, note that the dispersion relations for right- and left-circular polarizations are equivalent, reflecting the time-reversal and inversion symmetries of the B-phase. The significance of eq.(49) is that $\mathbf{D}(q,\omega) = 0$ determines the excitation energy of the uncoupled $J = 2^-$, $M = \pm 1$ modes, including dispersion and Fermi-liquid corrections. This dispersion relation reduces to the known limits for $q \longrightarrow 0$, [31]

$$\Big(1+\lambda(\omega)\frac{F^s_2}{5}\Big)\Big(\omega^2 - \frac{12}{5}\Delta^2\Big) - \frac{2F^s_2}{25}\lambda(\omega)\omega^2 = 0\ , \tag{50}$$

and for $F^s_2 = 0$,

$$\omega^2 = \frac{12}{5}\Delta^2 + c^2_{21}q^2\ , \tag{51}$$

with $c_{21} \simeq 0.4v_f$. [32]

Consider the dispersion relation for transverse current in the normal state. We set $\lambda = 0$, and $\xi_1$ reduces to

$$\xi_1 = \frac{15}{2}\ s\ \int_{-1}^{+1}\frac{dx}{2}\ \frac{x^2(1-x^2)}{s-x}\ , \tag{52}$$

with $s = \frac{\omega}{qv_f}$, giving the transcendental equation for the dispersion relation for transverse sound, and the condition, $F^s_1 + 3F^s_2/(1+F^s_1/3) > 6$, required for a propagating mode. [33] The closed form solution,

$$\frac{\omega}{q} = \sqrt{\frac{F^s_1}{15}}\ v_f \tag{53}$$

is obtained in the extreme limit, $F^s_1/15 \gg 1$, where we can carry out an expansion in $1/s$. This result cannot be used for quantitative estimates of the mode velocity since $F^s_1 \leq 15$ over the full pressure range of liquid $^3$He. However, the inequality which determines whether or not transverse sound is a propagating mode is satisfied above $p \simeq 3$ bar.

In superfluid $^3$He-B the collective mode contribution to eq. (48) drops out at low frequencies, $\omega \ll \Delta$. And, the quasiparticle contribution to the restoring force is also diminished by the gap in the spectrum. As a result, the transverse sound mode dies out rapidly at low temperature. To discuss the qualitative features of the transverse current mode, it is useful to consider the limit of very large $F^s_1$, where we can make a long wavelength expansion, $qv_f \ll \omega$; for convenience we also neglect $F^s_2$ for this discussion (this is a reasonable approximation since $F^s_2 \leq 1$). In this limit eq.(48) becomes

$$\Big(\frac{\omega}{qv_f}\Big)^2 = \frac{F^s_1}{15}\ [1-\lambda(\omega;T)]\ +\ \frac{2F^s_1}{75}\ \lambda(\omega;T)\ \frac{\omega^2}{(\omega+i\Gamma)^2 - \frac{12}{5}\Delta^2 - \frac{2}{5}q^2v_f^2}\ , \tag{54}$$

which clearly shows the resonant coupling to the $J = 2^-$, $|M| = 1$ mode. We include a phenomenological linewidth $\Gamma(T)$ for the $J = 2^-$ mode due to scattering of thermally excited quasiparticles. An approximate form for this linewidth is obtained from Einzel's analysis [34] of the measured $J = 2^+$ mode linewidth, [35] $\Gamma(T) \simeq \Gamma_c\sqrt{T/T_c}e^{-\Delta(T)/T}$, with $\Gamma_c \sim 10^6 - 10^7$ sec$^{-1}$ depending on pressure. Since the coupling to the collective mode drops out at intermediate temperatures and low frequencies, we obtain the limiting form for the dispersion relation,

$$\left(\frac{\omega}{qv_f}\right)^2 = \frac{F_1^s}{15}\, Y(T)\ , \tag{55}$$

originally obtained by Leggett [9] and Maki. [10] This is the long-wavelength limit of the normal-state dispersion relation with an effective $F_1^s$ reduced by Yosida function, $Y(T) = 1 - \lambda(0;T)$. The condition for an undamped propagating transverse mode at low frequencies is also obtained from eq.(48) by neglecting corrections of order $\left(\frac{\omega}{\Delta}\right)$, but retaining all terms orders in $\left(\frac{qv_f}{\omega}\right)$. This limit was analyzed by Combescot and Combescot, [11] where the condition, $T_L/T_c = 1 - 0.031(1 - 6/F_1^s)^2$, for the onset of Landau damping was obtained. The temperature below which the mode is overdamped by particle-hole excitations is more difficult to calculate. The authors of ref.( [11]) argue that this condition is given by the weaker condition, $F_1^s Y(T_o) \simeq 6$, which implies a weakly damped mode for $T > T_o \simeq 0.8\ T_c$ at $p = 15$ bar. Slightly higher frequencies are detrimental to transverse sound because of the off-resonant coupling to the order parameter mode. Equation (54) implies that for $\omega \lesssim \sqrt{\frac{12}{5}}\Delta$ the phase velocity of a transverse wave is reduced by level repulsion from the higher frequency $J = 2^-$ mode. The leading correction in $\frac{\omega}{\Delta}$ to the phase velocity in the long-wavelength limit is

$$\left(\frac{\omega}{qv_f}\right)^2 = \frac{F_1^s}{15}\, Y(T) - \frac{F_1^s}{90}(1 - Y(T))\left(\frac{\omega}{\Delta(T)}\right)^2\ . \tag{56}$$

Thus, one expects low-frequency transverse sound to disappear rapidly below $T_c$.

At higher frequencies the $J = 2^-$ mode can play a different role; the order parameter mode leads to a propagating transverse current wave at frequencies, $\sqrt{\frac{12}{5}}\Delta(T) \lesssim \omega < 2\Delta(T)$, and low temperatures. This fact is qualitatively clear from eq.(54). At low temperatures, the quasiparticle contribution to the restoring force, $\frac{F_1^s}{15}(1-\lambda)$, becomes small, or unsupportive of a propagating mode; but, for $\omega > \sqrt{\frac{12}{5}}\Delta(T)$, the restoring force from the off-resonant order parameter mode supports a propagating transverse current. Furthermore, this restoring force builds up as the temperature is lowered. The result is a propagating transverse current, with a reasonably large phase velocity, $\frac{\omega}{q} > v_f$, supported by the order parameter mode.

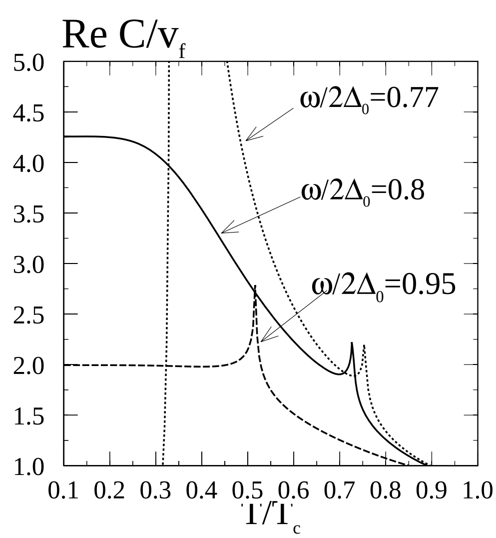


Fig. 1 Real part of the phase velocity from eq.(54) for transverse waves with $F_1^s = 15$ and $F_2^s = 0$. At low temperatures transverse waves propagate with a phase velocity $Re\ C > v_f$ for frequencies $0.77 \lesssim \frac{\omega}{2\Delta_0} < 1$.

The solution, $(\omega/q) = C(\omega, T)$, to the dispersion relation (eq.(48) or (54)) is in general complex. The phase velocity, defined by the real part of $C(\omega, T)$, is shown in Fig.(1) for several frequencies. For $\omega < \sqrt{\frac{12}{5}}\Delta_0$ there is no propagating solution below a temperature $T_2$, which is given approximately by $\sqrt{\frac{12}{5}}\Delta(T_2) = \omega$, as previously found by Maki and Ebisawa. [13] However, for $\omega > \sqrt{\frac{12}{5}}\Delta_0$ a propagating mode exists at $T = 0$ with a phase velocity that is largest near the collective mode frequency (compare the curves for $\omega = 0.95\Delta_0$ and $\omega = 0.8\Delta_0$). The group velocity is shown in Fig.(2) for the same frequencies, and as expected, it shows the dispersion characteristic of the coupling to the order parameter mode.

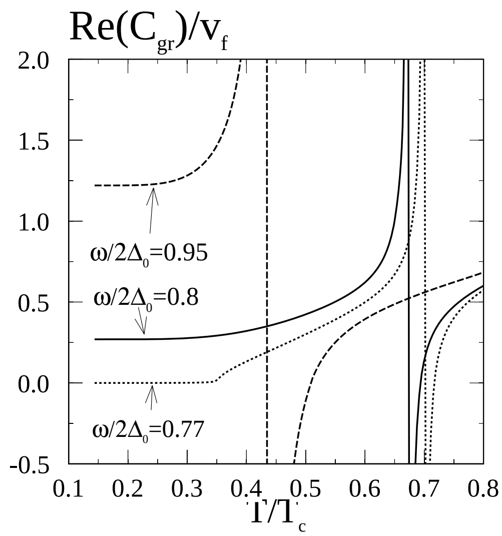


Fig. 2 The group velocity for transverse waves with $F_1^s = 15$ and $F_2^s = 0$. Both the group velocity and phase velocity are strong functions of frequency and temperature, indicative of the frequency response of the $J = 2^-$ mode.

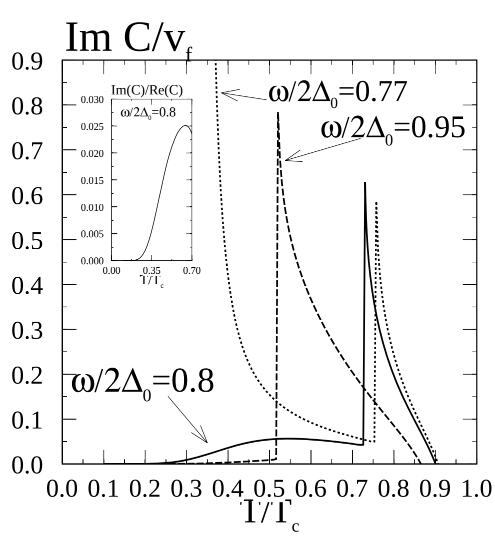


Fig. 3 The imaginary part of the complex phase velocity for $F_1^s = 15$ and $F_2^s = 0$. The sharp structure at high temperatures is due to pair-breaking; the onset is at $\omega = 2\Delta(T_{pb})$. The inset shows the ratio of the imaginary and real parts for $\frac{\omega}{2\Delta_0} = 0.8$. At low temperature the attenuation drops rapidly as the thermally excited quasiparticles freeze out.

The damping of the transverse current mode also increases as $\omega$ approaches the collective mode frequency, as shown in Fig.(3). However, the frequency range of strong quasiparticle damping is $|\omega - \sqrt{\frac{12}{5}}\Delta(T)| < \Gamma(T) \propto e^{-\Delta/T}$. The inset of Fig.(3) shows the ratio of the imaginary and real parts of the complex phase velocity for temperature below the pair-breaking edge, *i.e.* $\omega < 2\Delta(T)$. Thus, the propagation of transverse current via the order parameter collective mode appears feasible at very low temperatures. Note also that the pair-breaking threshold is evident in both the real and imaginary parts, but most strikingly in the imaginary part of the complex phase velocity as an abrupt increase in damping for $\omega > 2\Delta(T)$.

The calculations based on the long-wavelength limit of the dispersion relation are only qualitatively correct solutions to the full $\omega$- and $q$-dependent dispersion relation, eq.(48). The full dispersion equation requires the $q$-dependent response function $\lambda(\eta,\omega)$ given in the appendix. Figure (4) shows a comparison between the low-temperature limit of the phase velocities calculated from eq.(48) and the long-wavelength approximation in eq.(54); there are corrections of $\lesssim 5$ % to the phase velocity over the important temperature range below the pair-breaking edge. Short-wavelength corrections are more important at frequencies and temperatures near the pair-breaking edge, where the singularity at $\omega = 2\Delta(T_{pb})$ is smoothed out by dispersion effects. Also note that the phase velocity is sensitive to $F_2^s$ because this parameter tunes the frequency of the $J = 2^-$ mode (see inset of Fig.(4)).

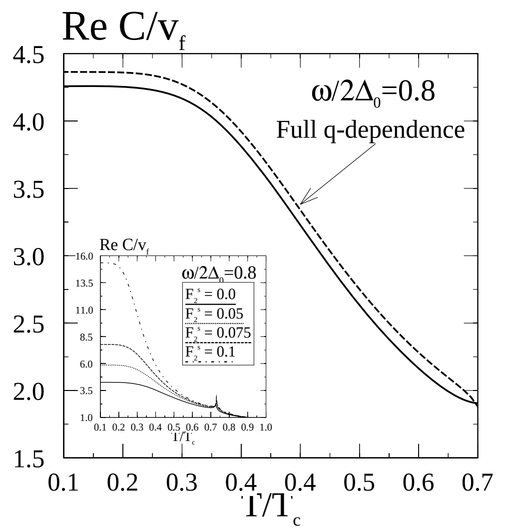


Fig. 4 Short-wavelength corrections to $C/v_f$. The dashed curve corresponds to the real part of the phase velocity calculated from the complete dispersion equation (48) for $F_1^s = 15$ and $F_2^s = 0$. The solid curve is the corresponding phase velocity calculated in the long-wavelength approximation from eq.(54). The inset shows the variation of *Re* $C/v_f$ with $F_2^s$.

### B. Longitudinal Sound

High-frequency transverse currents propagate at low temperatures because of the coupling to an order parameter collective mode. Essentially this same phenomena is responsible for the survival of longitudinal sound in the superfluid at low temperatures. In that case the phase mode, a Goldstone mode with a linear dispersion relation, $\omega = \frac{v_f}{\sqrt{3}}q$, couples strongly to density and longitudinal current, and leads to propagating longitudinal sound with a phonon dispersion relation; the coupling to density and longitudinal current pushes the velocity of the coupled mode up to the hydrodynamic sound velocity, $c_1$. [28] The cross-over from zero sound, suppported by the quasiparticle contribution to the molecular field at $T = T_c$, to collisionless sound in the superfluid, supported by the phase mode at $T = 0$, was discussed early on by Leggett, [9] who considered the long wavelength collective modes in a neutral, isotropic Fermi superfluid. For weak Fermi-liquid interactions, $F_{0,1}^s \gtrsim 0$, the longitudinal sound mode exists above $T_c$, but becomes overdamped below $T_c$ as the gap in the quasiparticle spectrum develops. However, at sufficiently low temperatures longitudinal sound reappears due to the coupling to the order parameter phase mode. For stronger molecular field interactions, $F_{0,1}^s \gg 1$, as in $^3$He, the longitudinal mode does not become overdamped by reduction of the quasiparticle restoring force. Thus, the basic physics underlying the survival of both longitudinal and transverse current modes in superfluid $^3$He-B is similar; a reduction in the restoring force from quasiparticles as the gap develops, followed by propagation at low temperatures as a consequence of coupling to an order parameter collective mode, for frequencies above the (uncoupled) order parameter mode frequency. Of course the unique feature of $^3$He-B is the existence of order parameter modes that couple to transverse current. And since these modes are not Goldstone modes, the frequency and temperature dependences of the phase velocity are considerably different than those for the longitudinal current.

For $^3$He-B these general features of longitudinal sound follow from the dispersion relation,

$$\left(\frac{\omega}{q}\right)^2 = c_1^2 \left\{ 1 + \frac{4}{15}\left(\frac{qv_f}{\omega}\right)^2 (1-\lambda(\omega)) + \frac{8}{75}\left(\frac{qv_f}{\omega}\right)^2 \lambda(\omega) \left[\frac{\omega^2}{(\omega+i\Gamma)^2 - \frac{12}{5}\Delta^2 - \frac{7}{15}(qv_f)^2}\right]\right\} , \tag{57}$$

which can be derived in analogy to eq.(54) for the transverse current (*c.f.* ref.[ [21]]). We have set $F^s_{1,2} = 0$ to simplify the formula; corrections from the higher Landau parameters are easily included. This dispersion relation shows contributions to the restoring force from the coupling to the phase mode (first term) and the quasiparticles (second term). In addition, there is a coupling to the high-frequency $J = 2^-$ mode (third term). In the low-frequency limit, $\omega \ll \Delta$, the $J = 2^-$ mode drops out, and we obtain the collisionless sound velocity,

$$\frac{c^2 - c_1^2}{c_1^2} \simeq \frac{4}{5}\frac{Y(T)}{1+F_0^s} . \tag{58}$$

This formula agrees with the well-known result for the zero sound velocity in the normal state, and reduces to the hydrodynamic sound velocity for $T \to 0$, when there are no excitations other than the phase mode providing the restoring force. At higher frequencies the $J = 2^-$ mode contributes; the coupling is of order $(\frac{qv_f}{\omega})^2$ for frequencies off resonance. However, near resonance this coupling leads to strong damping of longitudinal sound. A significant difference between the coupling of the $J = 2^-$ modes to the longitudinal and transverse currents is that the longitudinal current couples to the $J = 2^-$, $M = 0$ mode, while the transverse current couples to the $M = \pm 1$ modes. Thus, a weak ($\gamma H \ll \Delta_0$) longitudinal magnetic field, $\vec{H}||\vec{q}$, has essentially no effect on the propagation of longitudinal sound; the situation is more interesting for the transverse modes.

### C. Magnetic Field Effect

The phase velocity for right- (RCP) and left-circularly polarized (LCP) waves is identical in zero field, reflecting the time-reversal and reflection symmetries of the B-phase order parameter. A magnetic field with $\vec{H}||\vec{q}$ breaks time-reversal symmetry, and lifts the degeneracy of the two polarizations. The origin of this effect is the Zeeman splitting of the $J = 2^-$ modes. [36] In a magnetic field these modes split into a five-fold multiplet,

$$\omega_M = \omega_0 + M\ g_{2-}\ \omega_L\ , \qquad M = 0, \pm 1, \pm 2 \qquad ; \quad \omega_L = \frac{\gamma H}{[1 + \frac{1}{3}F_0^a(2+Y(T))]} , \tag{59}$$

where $\omega_L$ is the effective Larmor frequency and $g_{2-}$ is the Landé g-factor for the $J = 2^-$ modes. [37] Since the field does not break the cylindrical symmetry about the propagation direction, it is still the case that only the $M = \pm 1$ modes couple to transverse current. The resulting dispersion relations for the two polarizations are again given by eq.(48), but with the Zeeman splitting included in the dispersion relations (eqs.(48)-(54)) of the $J = 2^-$, $M = \pm 1$ modes. In the zero temperature, long-wavelength limit, the splitting in the phase velocity for RCP and LCP waves is real, and calculated to be,

$$\frac{C_+ - C_-}{C} \simeq \frac{4F_1^s}{75}\lambda(\omega)\left(\frac{g_{2-}\ \omega_L}{\omega}\right)$$
$$\div\left[C(\omega)^2\left(\frac{\omega^2 - \frac{12}{5}\Delta^2}{\omega^2}\right)^2 + \frac{F_1^s}{25}\lambda(\omega)\left\{\frac{\omega}{\lambda}\frac{d\lambda}{d\omega}\left(\frac{\omega^2-4\Delta^2}{\omega^2}\right)\left(\frac{\omega^2 - \frac{12}{5}\Delta^2}{\omega^2}\right) + \frac{2}{5}\left(\frac{4\Delta^2}{\omega^2}\right)\right\}\right] , \tag{60}$$

for $\omega_L \ll \Delta$. Near the collective mode resonance the splitting reduces to that of the $J = 2^-$ modes,

$$\frac{C_+ - C_-}{C} \to \left(\frac{2g_{2-}\omega_L}{\omega}\right) , \qquad |\omega^2 - \frac{12}{5}\Delta^2| \ll \omega^2 . \tag{61}$$

This splitting leads to a rotation of the plane of polarization of a linearly polarized wave. At low, but finite temperatures, the imaginary parts of the complex phase velocities also split in a field, giving rise to the analog of circular dichroism of electromagnetic waves in magnetic media.

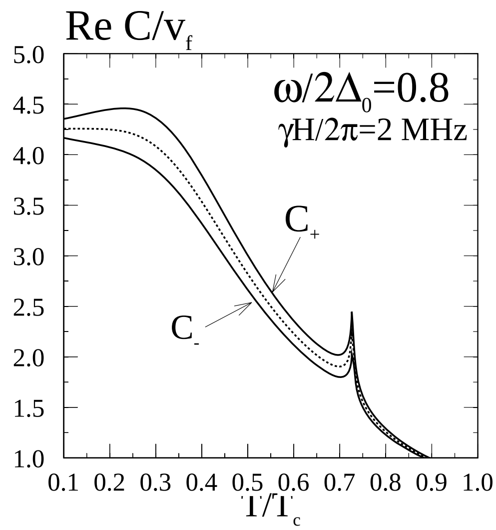


Fig. 5 Magnetic field effect on the phase velocity of right- and left-circularly polarized waves. The real part of $C_{\pm}(\omega, T)$ for transverse waves with polarizations $\hat{e}^{(\pm)}$ are shown for a field of $\frac{\gamma H}{2\pi} \simeq 2$ MHz ($\omega_L/2\Delta_0 \simeq 0.05$). We have purposely chosen a large field to emphasize the splitting.

For linearly polarized waves, in the low temperature limit, the rotation of the plane of polarization has a spatial period $\lambda_H \simeq 2\pi \frac{C}{g_{2-}\ \omega_L}$, where $C$ is the zero-field phase velocity. When $\lambda_H$ becomes comparable to or smaller than the path length for a propagating transverse wave, the magnetic field will have a dramatic effect on the response of an incident linearly polarized wave. For $\omega_L/2\pi = 1.3$ MHz ($H \sim 220$ G) and $C \sim v_f \simeq 32$ m/s, the wavelength for the rotation of the polarization is $\lambda_H \simeq 0.25$ mm. This field dependence and long length scale for the rotation of the polarization suggests a novel way of detecting a propagating transverse current mode in $^3$He-B. Orthogonal polarization transducers could be used to excite and detect transverse current waves. When the field is increased so that the path length is $d = \frac{1}{4}\lambda_H$ the polarization will have rotated by $\pi/2$. Such a signal would be direct evidence for the propagation of transverse currents via the $J = 2^-$, $M = \pm 1$ order parameter modes, and would not require disentangling from an incoherent background that has plagued the detection of transverse sound in normal $^3$He. In addition to the rotation of the polarization, the weak quasiparticle damping at low temperatures leads to a differential attenuation of the RCP and LCP waves; thus, initially linearly polarized waves will become elliptically polarized rather than simply having the linear polarization rotated as the wave propagates along the field.

### D. Conclusions

Finally, it is important to note that the calculations described here correspond to the response of bulk $^3$He-B to a probe of transverse excitations. In the high attenuation regime, where the transverse current decays rapidly with distance from the moving surface, the acoustic impedance is likely to have substantial contributions from surface effects not included in any present theory. Thus, the sharp structures shown in Figs. (1) and (2) at the pair-breaking edge and at the $J = 2^-$ mode for $\omega < \sqrt{\frac{12}{5}}\Delta_o$ may be only qualitatively representative of the frequency and temperature dependences of the transverse acoustic impedance. However, in the low attenuation regime transverse current will propagate and the predicted frequency and temperature dependences of the phase and group velocities should be observable. Nevertheless, there may be additional channels for coupling of transverse momentum from the vibrating surface to liquid $^3$He. A microscopic theory of the transverse acoustic impedance for high frequencies does not presently exist, but will necessarily involve the order parameter modes, including the influence of the surface, as well as low-energy quasiparticle states associated with the deformed order parameter near the surface.

In summary, transverse waves in $^3$He-B at very low temperatures are expected to propagate as a consequence of the off-resonant coupling of the current to the $J = 2^-$, $M = \pm 1$ order parameter collective modes. The effects of a parallel field, $\vec{H}||\vec{q}$, on the polarization of transverse current waves are analogous to those of circular dichroism and birefringence of electromagnetic waves in superconductors, [38] and may provide a unique way in which to detect propagating transverse currents in superfluid $^3$He-B.

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[38] We note that circular dichroism and birefringence of transverse current modes in $^3$He-A is expected to exist even in zero magnetic field as consequence of the spontaneously broken time-reversal and space parity of the ABM order parameter. [30] However, the origin of the zero-field effect on the transverse current modes in $^3$He-A is quite different, and its order of magnitude much smaller than the field-induced effect in $^3$He-B.

## IV. APPENDIX

The response function,

$$\lambda(\eta,\omega) = \Delta^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \frac{\left(\eta^2 + 2\epsilon\omega\right)\beta(\epsilon+\omega/2) + \left(\eta^2 - 2\epsilon\omega\right)\beta(\epsilon-\omega/2)}{(4\epsilon^2-\eta^2)(\omega^2-\eta^2)+4\eta^2\Delta^2} \ , \tag{62}$$

with $\beta(\epsilon) = 2\pi i \frac{\Theta\left(\epsilon^2-\Delta^2\right)}{\sqrt{\epsilon^2-\Delta^2}} \tanh\frac{\epsilon}{2T}$ and $\eta = \vec{q}\cdot\vec{v}_{\hat{p}}$, appears in connection with nearly all high-frequency excitations of superfluid $^3$He-B. Qualitatively, $\Re\mathrm{e}(\lambda)$ measures the *stiffness* of the condensate at temperature $T$, frequency $\omega$ and wavevector $\vec{q}$. For many purposes, the long-wavelength limit is adequate; $\lambda(\omega)$ is then independent of the direction $\hat{p}$ in momentum space,

$$\lambda(\omega) = \Delta^2 \int_{\Delta}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon^2-\Delta^2}} \left\{ \frac{\tanh(\epsilon/2T)}{\epsilon^2 - \frac{1}{4}\omega^2} \right\} \ , \tag{63}$$

which is real and of order one for frequencies below the pair-breaking edge, except for $T \to T_c$ or $\omega \to 2\Delta(T)$. Near the pair-breaking edge $\lambda(\omega)$ has a weak singularity, $\lambda \sim \frac{1}{\sqrt{2\Delta-\omega}}$. Above the pair-breaking edge, $\omega > 2\Delta(T)$, the response function acquires an imaginary part reflecting the density of broken pair excitations,

$$\Im\mathrm{m}(\lambda) = \frac{\pi}{2}\left(\frac{2\Delta}{\omega}\right)\left\{ \frac{\tanh(\omega/4T)}{\sqrt{(\omega/2\Delta)^2-1}} \right\} \qquad ; \qquad \omega > 2\Delta \ . \tag{64}$$

For transverse sound the long-wavelength approximation is not valid, so it is important to examine the full $\vec{q}$-dependent function. A shift of the integration variable gives,

$$\lambda \ = 4\Delta^2 \int_{\Delta}^{\infty} d\epsilon \frac{\tanh\frac{\beta\epsilon}{2}}{\sqrt{\epsilon^2-\Delta^2}} \tag{65}$$

$$\times \frac{(\omega^2-\eta^2)[4\epsilon^2(\omega^2+\eta^2) - 4\eta^2\Delta^2 - (\omega^2-\eta^2)^2]}{\left[(4(\epsilon-\frac{\omega}{2})^2-\eta^2)(\omega^2-\eta^2)+4\eta^2\Delta^2\right]\left[(4(\epsilon+\frac{\omega}{2})^2-\eta^2)(\omega^2-\eta^2)+4\eta^2\Delta^2\right]} \ , \tag{66}$$

(67)

which can be transformed to

$$\lambda = \frac{\Delta^2}{2} \int_0^{\infty} d\xi \ T \sum_{\epsilon_n} \frac{1}{\epsilon_n^2+\xi^2+\Delta^2} \frac{[4\xi^2(\omega^2+\eta^2)+4\omega^2\Delta^2-(\omega^2-\eta^2)^2]}{(\omega^2-\eta^2)(\xi^2-\Omega_+^2)(\xi^2-\Omega_-^2)}, \tag{68}$$

with $\Omega_\pm^2 = \left[\omega^4 - \eta^4 - 4\Delta^2\omega^2 \pm 2\omega\eta\sqrt{(\omega^2-\eta^2)(\omega^2-\eta^2-4\Delta^2)}\right] / \left[4(\omega^2-\eta^2)\right]$, by the change of variables, $\epsilon = \sqrt{\xi^2+\Delta^2}$, and the series representation, $\tanh(\frac{\beta\epsilon}{2})/2\epsilon = T\sum\frac{1}{\epsilon_n^2+\epsilon^2}$. The $\xi$-integral is easily performed and the resulting expression for $\lambda(q,\omega)$ is calculated by summing over Matsubara frequencies to the required precision.